A new test for chaotic dynamics using lyapunov exponents

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ABSTRACT

We propose a new test to detect chaotic dynamics, based on the stability of the largest Lyapunov exponent from different sample sizes. This test is applied to the data used in the single-blind controlled competition tests for non-linearity and chaos that were generated by Barnett *et al.* (1997), as well as to several other chaotic series. The results suggest that the new test is particularly effective when compared to other stochastic alternatives (both linear and nonlinear). The test size is one for large samples, although for small sample sizes it diminishes below the nominal size for two out of the three chaotic processes considered.

JEL classification numbers: C13, C14, C15, C22

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1. Introduction

In a dissipative dynamical system, the existence of a positive Lyapunov exponent is usually taken as an indication of the chaotic character of the system. Lyapunov exponents provide information on the intrinsic instability of the trajectories of the system, and are computed as the average rate of exponential convergence or divergence of trajectories that are near each other in the phase space.

In recent years, there has been a burgeoning literature on the calculation of Lyapunov exponents for an unknown dynamical system reconstructed from a single time series. The seminal paper of Wolf *et al.* (1985) provides an algorithm to compute Lyapunov exponents in empirical applications, but this is sensitive to both the number of observations and the degree of noise in the data. More recently, however, some authors have proposed new methods for estimating Lyapunov exponents that perform well even for small samples [see, among others, Dechert and Gençay (1992), Abarbanel *et al*. (1991, 1992), and Rosenstein *et al*. (1993)].

There are many papers that use Lyapunov exponents to detect chaotic dynamics in financial time series, especially in exchange rate series. Early examples of research in this area include Bajo-Rubio *et al.* (1992) and Dechert and Gençay (1992), where Lyapunov exponents are used to distinguish between linear, deterministic processes (with negative Lyapunov exponents) and non-linear, chaotic deterministic processes (where the largest Lyapunov exponent is positive). These and other papers have been criticised for the absence of a distributional theory providing a statistical framework for hypothesis testing using the Lyapunov exponents that are calculated. In this sense, Gençay (1996) presents a methodology to compute the empirical distributions of Lyapunov exponents using a blockwise bootstrap technique. This methodology provides a formal test of the hypothesis that the largest Lyapunov exponent equals some hypothesised value, and can be used to test for chaotic dynamics. The test proposed by Gençay (1996) is particularly useful in those cases where the largest Lyapunov exponent is positive, but very close to zero. More recently, Bask and Gençay (1998) use the same statistical framework to provide a test for the presence of a positive Lyapunov exponent in atime series used. The numerical examples show that both the Gençay (1996) and the Bask and Gençay (1998) test statistics behave well for small samples. These papers have been very influential from an empirical point of view and, for example, Bask (1998), using the test suggested by Bask and Gençay (1998), finds evidence that some exchange rates can be characterised by deterministic chaos.

Despite the growing interest in the econometric literature aimed at distinguishing between non-linear deterministic processes and non-linear stochastic processes, there remain important disagreements and controversy about the results. A key paper in this area is Barnett *et al.* (1997), where some data series were simulated from different generating models in order to evaluate the behaviour, both for large and small samples, of five highly regarded tests for non-linearity or chaos. The tests considered in that paper are the Hinich bispectral test (Hinich, 1982), the BDS test (Brock *et al*., 1996), the NEGM test (Nychka *et al*., 1992), the White test (White, 1989), and the Kaplan test (Kaplan, 1994). The results concerning the power function of some of these tests proved to be rather surprising, since none of them had the ability to isolate the origins of the non-linearity or chaos to within the structure of the economy.

The aim of this paper is to propose a new test for the presence of chaos, based on the behaviour of the estimated Lyapunov exponents, for different sample sizes. As we shall try to illustrate, while the largest exponent of a chaotic process is more or less stable with respect to sample size (it displays stationary behaviour when the sample size increases), the largest Lyapunov exponent of a stochastic process is not. Therefore, we suggest testing chaotic dynamics by estimating the empirical distributions of the largest Lyapunov exponents for different sample sizes and comparing their means. The proposed new test proves to be very effective when compared to stochastic processes, hence providing further refinement over those of Gençay (1996) and Bask and Gençay (1998).

The rest of the paper is organised as follows. Section 2 presents the statistical framework used in the paper. Section 3 discusses the stability of the largest Lyapunov exponent with respect to sample size. Section 4 proposes the new test for distinguishing chaos from random behaviour. Section 5 reports the results of applying our test to several chaotic processes, as well as to the simulated data used in the single-blind controlled competition tests performed by Barnett *et al.* (1997). On the other hand we have also tested for chaos in several exchange rate time series. Finally, Section 6 provides some concluding remarks.

2. A statistical framework for testing chaotic dynamics via Lyapunov exponents

In order to examine the properties of a deterministic dynamical system we make use of ergodic theory, since it provides a statistical framework where different degrees of the complexity of attractors and motions can be distinguished [see Eckmann and Ruelle (1985) for a survey]. Furthermore, ergodic theory allows us to describe the time averages of a dynamical system and to consider when transients become irrelevant. Once transients are completed, the motion of the dynamical system typically settles near a subset of $\Re^n$, called an *attractor*. In the particular case of dissipative systems, where the phase-space volumes are concentrated by time evolution, the volume occupied by the attractor is in general very small in relation to the phase space. Even if a system's volume contracts, it does not mean that its length is contracted in all directions: some directions may be stretched and some directions contracted. This implies that, even in a dissipative system, the final motions may be unstable within the attractor. This instability usually manifests itself in *sensitive dependence on initial conditions*, which means an exponential separation of orbits (as time goes on) of points that were initially very close each other on the attractor. In this case, we say that the attractor is a *strange attractor* and that the system is *chaotic*.

In ergodic theory, statistical averages can be computed either in terms of time averages or space averages. Let us consider, for simplicity, a discrete dynamical system of dimension n, $\vec{x}_{t+1} = \vec{F}(\vec{x}_t)$, where $\vec{F} : \Re^n \longrightarrow \Re^n$ is a differentiable vectorial function. The *time average* of a function φ along a (forward) trajectory $\vec{x}_i$ with initial condition $\vec{x}_0$, of a discrete dynamical system is defined by

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N-1} \varphi(\vec{x}_i)$$

In a similar way for a continuous flow ϕ_t, arising from a continuous dynamical system $\frac{d\vec{x}}{dt} = \vec{F}(\vec{x}),$ the time average of a function along a (forward) trajectory is

$$\lim_{T\to\infty}\frac{1}{T}\int_0^T \varphi(\phi_t(x))dt$$

Time averages often depend on initial conditions. However, when the dynamical system has an attractor, all trajectories have the same statistical properties.

A measure of complexity in chaotic motion may be obtained by analysing the sensitivity of the dynamical behaviour to the initial conditions given by two infinitely close initial states. For chaotic systems, points in a common neighbourhood in the phase space separate exponentially with time. Let us illustrate the basic idea by means of a discrete dynamical system of dimension *n*, $\vec{x}_{t+1} = \vec{F}(\vec{x}_t)$. In order to examine the stability of the trajectories of the system, let us consider how the system amplifies a small difference between the initial conditions $\vec{x}_0$ and $\vec{x}'_0$:

$$\vec{x}_T - \vec{x}'_T = \vec{F}^T(\vec{x}_0) - \vec{F}^T(\vec{x}'_0) \cong D\vec{F}^T(\vec{x}_0)(\vec{x}_0 - \vec{x}'_0)$$

where $\vec{F}^T(\vec{x}_0) = \vec{F}(\vec{F}(...\vec{F}(\vec{x}_0)...))$ denotes the *T* successive iterations of the dynamical system starting from the initial condition $\vec{x}_0$, and where $D\vec{F}^T(\vec{x}_0)$ is the Jacobian of the function $\vec{F}(\vec{x})$.

By the chain rule, we have

$$D\vec{F}^T(\vec{x}_0) = D\vec{F}(\vec{x}_{T-1})D\vec{F}(\vec{x}_{T-2})..........D\vec{F}(\vec{x}_0)$$

In this context, the Lyapunov exponents are defined as follows (Guckenheimer and Holmes, 1990): Let us consider the family of subspaces $V_i^{(1)} \supset V_i^{(2)} \supset ..... \supset V_i^{(n)}$ in the tangent space at $\vec{F}^i(\vec{x})$ and the numbers $\lambda_1 \geq \lambda_2 \geq ..... \geq \lambda_n$ with the properties that:

(1) $D\vec{F}(V_i^{(j)}) = V_{i+1}^{(j)}$

(2) $\dim V_i^{(j)} = n + 1 - j$

(3) $\lim_{T\to\infty}\frac{1}{T}\ln\|\sqrt{(D\vec{F}^T)^* \cdot (D\vec{F}^T)}(\vec{x}_0)\| = \lambda_j$ for all $\vec{x}_0 \in V_0^{(j)} - V_0^{(j-1)}$, where $(D\vec{F}^T)^*$ is the transpose of $D\vec{F}^T$

Then, the real numbers λ_j are called the *Lyapunov exponents* of $\vec{F}$ at $\vec{x}_0$. Lyapunov exponents offer information on how orbits on the attractor move apart (or together) given the dynamic evolution of the system. One can also define them by the rate of stretching or shrinking of line segments, areas, and various dimensional sub-volumes in the phase space. Line segments grow or shrink by a factor of $e^{t\lambda_1}$, areas by a factor of $e^{t(\lambda_1+\lambda_2)}$ and so forth. If one or more of the Lyapunov exponents are positive, then we have *chaos* in the motion of the system. The sum of the Lyapunov exponents is negative $(\lambda_1 + \lambda_2 + ... + \lambda_n < 0)$ for dissipative systems [see Abarbanel (1996)].

The possibility of obtaining, in a deterministic dynamical system, Lyapunov exponents that are representative of short-run divergences in trajectories with many closed initial points is based on Oseledec's (1968) *multiplicative ergodic theorem*. If we assume that there exists an ergodic measure of the system, this theorem justifies the use of arbitrary phase space directions when calculating the largest Lyapunov exponent. In this case, the Lyapunov exponents have a mean in a global sense, allowing the complexity of a deterministic dynamical system of dimension *n* to be characterised simply by *n* real numbers.

Oseledec's (1968) multiplicative ergodic theorem states that, under quite general conditions on the function $\vec{F}$, the limit of expression (3) does exist for almost all $\vec{x}_0$ (with respect to the invariant measure μ) and is independent of the initial condition $\vec{x}_0$ considered (except for a set of null measure). Therefore, the multiplicative ergodic theorem implies that the Lyapunov exponents are invariant numbers representing "globally" the complexity of the dynamical system under study, independently of the initial condition considered. In this sense it is also important to point out that Gençay and Dechert (1992), Gençay and Dechert (1996) and Dechert and Gençay (2000), have studied the topological invariance of the Lyapunov exponent estimator from the observed dynamics.

Oseledec's theorem is based on the ergodic theory of deterministic dynamical systems and justifies the use of arbitrary phase space directions when calculating the largest Lyapunov exponents. Nevertheless, as both Whang and Linton (1999) and Tong (1990) point out, Lyapunov exponents can be interpreted within the standard non-linear time series framework as a measure of local stability and is of interest even without any direct connection with deterministic chaos.

There are several suitable estimation methods to obtain Lyapunov exponents based on kernels, nearest neighbours, splines, local polynomials and neural nets [see Härdle and Linton (1994) for a general discussion]. McCaffrey *et al.* (1992) distinguish two classes of methods for estimating the largest Lyapunov exponent λ_{max}: (i) Direct methods like Wolf *et al.* (1985) or Rosenstein *et al.* (1993), which assume that the initial divergence $(\vec{x}_0 - \vec{x}'_0)$ grows at the exponential rate given by λ_{max} in the reconstructed state space of a time series ; and (ii) Jacobian methods, where data are used to estimate the Jacobians from an estimation of the conditional expectation of the process, which allows λ_{max} to be estimated. Examples of Jacobian methods are those proposed by MacCaffrey *et al.* (1992), Nychka *et al.* (1992) or Gençay (1996). Although both conceptions (direct and Jacobian methods) agree with respect to the conception of estimated Lyapunov exponents for chaotic nonlinear deterministic processes, their approach to the Lyapunov exponents of stochastic processes are extremely different.

On one hand, direct methods act directly on the time series to estimate λ_{max} and are not intended, *ex ante*, to separate the effect of the random variables into the series. Direct methods are based on the philosophy that chaos provides a link between determinism and randomness; in this view the dimension and λ_{max} of IID noise, in theory, is infinite, and if the deterministic definition of the Lyapunov exponents is taken literally, the Lyapunov exponents are also infinite in presence of noise. Schuster (1996, p. 112) or Eubank and Farmer (1990, p. 160) support this conception where the λ_{max} of stochastic processes will, necessarily, be positive because of the infinite dimensionality of the noise.

On the other hand, in order to estimate the Lyapunov exponents in stochastic processes, Jacobian methods use nonparametric regression tools (Kernels, neural networks, etc.) trying to isolate the deterministic conditional mean of the process and estimate the derivatives in order to reconstruct the Jacobian. In this conception stochastic processes may have a negative λ_{max}, for instance stationary linear autoregressions have $\lambda_{max} < 0$, while for unit root process $\lambda_{max} = 0$ and only for explosive linear autoregressions do we get $\lambda_{max} > 0$ (See Whang and Linton (1999), p. 5) .

Therefore, within the theory of dynamical systems, a chaotic system is characterised by globally bounded trajectories in the phase space with a positive largest Lyapunov exponent, while in the frame of direct methods to estimate λ_{max}, the largest Lyapunov exponent of a white noise process is (in theory) infinite [see Schuster (1996) or Eubank and Farmer (1990)].

Nevertheless, in practical implementations, using finite time series, any standard algorithm for calculating the largest Lyapunov exponent, with direct methods, will find a finite positive value for this exponent, for any random process. Therefore, the largest Lyapunov exponent on its own is not able to distinguish between a chaotic, non-linear deterministic process and a random process. This problem is especially relevant in financial time series, where non-linear stochastic processes, such as GARCH processes, are usually postulated as alternative models to the chaotic behaviour [see, e. g., Hsieh (1991)].

Gençay (1996) proposed a statistical framework for testing chaotic dynamics using a moving blocks bootstrap procedure.

Consider a sequence of weakly dependent stationary random variables $\{X_1, X_2, ..., X_N\}$, and let $\{x_1, x_2, ..., x_N\}$ be a time series realisation of this stochastic process. According to Künsch (1989) and Liu and Singh (1992), the distribution of certain estimators of interest can be consistently constructed by applying a moving blockwise bootstrap. Let $B_t^d = \{x_t, x_{t+1}, ..., x_{t+d-1}\}$ denote a moving block of d consecutive observations, where $t \leq N - d + 1$. For a time series of N elements, we can form a set $\{B_1^d, \ ... \ , B_{N-d+1}^d\}$ of blocks with length *d*. Let $k = int(N/d)$ [where *int()* denotes the integer part], so that by sampling with replacement of *k* blocks denoted by $\{B_{i_1}^d, \ .... \ , B_{i_k}^d\}$ we will form the bootstrap sample.

In order to obtain the sample distribution of the largest Lyapunov exponent λ_{max}, we will repeat this procedure to construct a sequence of sub-families of *k* blocks taken with replacement from the family of *d*-dimensional blocks $\{B_1^d, \ ... \ , B_{N-d+1}^d\}$ that can be generated with the time series $\{x_1, x_2, ..., x_N\}$. For each sub-family of *k* blocks, we can apply some standard procedure to compute the largest Lyapunov exponent $\tilde{\lambda}_{max}$ by taking the pairs of nearest neighbours from each sub-family of blocks. Repeating this process a large number of times, we will obtain the empirical distribution of the largest Lyapunov exponent $\tilde{\lambda}_{max}$.

At this point it is important to observe the difficulty of Jacobian methods for estimating λ_{max} within the framework of bootstrap replications. As Ziehmann *et al.* (1999) pointed out, a bootstrap algorithm must be used with caution if Lyapunov exponent estimates rely on the product of matrices because matrix multiplication does not commute, except in one dimension. In order to avoid such complications with the product of Jacobians along the trajectory, we use a simple direct method for estimating the largest Lyapunov exponent λ_{max} of a time series proposed by Rosenstein *et al.* (1993). Given that the divergence between the nearest neighbours takes place at a rate approximated by the largest Lyapunov exponent, Rosenstein *et al.* suggest the choice of a pair of neighbours as initial conditions for different trajectories, and to estimate λ_{max} by averaging the exponential divergences of these initially close state-space trajectories.

Rosenstein's method may be outlined as follows. Let us consider an observed time series $\{x_1,\ x_2,\ ...,\ x_N\}$. Following Takens' (1981) theorem, we start by reconstructing the phase-space vector using the blocks $\{B_1^d,....,B_{N-d+1}^d\}$ defined above, where $B_t^d = \{x_t,\ x_{t+1},\ ...,\ x_{t+d-1}\}$. For each point B_t^d, we search for the nearest neighbour point $B_{t^*}^d$ in the reconstructed phase space that minimises the distance to that reference point:

$$d_t(0) = \min_{B_{t^*}^d} \| B_t^d - B_{t^*}^d \| ,$$

where $\|...\|$ denotes the Euclidian norm. To consider each pair of neighbours as initial conditions for different trajectories, the temporal separation between them should be greater than the mean period of the time series:

$$| t - t^* | > mean\ period$$

This mean period can be estimated as the reciprocal of the mean frequency of the power spectrum of the time series under study.

The divergence between the nearest neighbours B_t^d and $B_{t^*}^d$ takes place at a rate approximated by the largest Lyapunov exponent:

$$d_t(i) \cong d_t(0)\ \ exp(\lambda_{max} i \Delta t)\ \ for\ \ i = 1,.....,T \tag{1}$$

where i is the number of discrete-time steps following the nearest neighbour, Δt is the sampling period of the time series, and $d_j(i)$ is the distance between the jth pair of nearest neighbours after i discrete time steps. (Recall that $i\Delta t$ corresponds to seconds so that Lyapunov exponents are expressed in bits/second). Taking the logarithm of both sides of this last expression, we obtain:

$$\log(d_t(i)) \cong \log(d_t(0)) + \lambda_{\max} i \Delta t \tag{2}$$

For each value of t between 1 and N-d+1, this equation represents a set of approximately parallel lines, each with a slope that is approximately proportional to λ_{max}. The largest Lyapunov exponent is then estimated using a least-square fit with a constant to the average line defined by $\langle \log(d_t(i) \rangle$, where $\langle\ \ \rangle$ denotes the average value over all values of t.

There exist several key parameters in the Rosenstein *et al.* algorithm. Besides the embedding dimension (d) (that will be called the moving-block length for the moving blocks bootstrap procedure), we have to select the lag or reconstruction delay, the mean period and the number of discrete-time steps (i) allowed for divergence between nearest neighbours B_t^d and $B_{t^*}^d$ in the phase space. We explain later how to select these parameters following the author's recommendations.

3. Stability of largest Lyapunov exponents with the sample size for chaotic processes

From a theoretical point of view, the reason for the stability of the largest Lyapunov exponent with respect to the sample size can be found in Oseledec's (1968) theorem, which states that for a large enough sample size, these exponents will converge to some stable values associated with the complexity of the attractor.

For chaotic time series, Oseledec's theorem guarantees the possibility of making short-run forecasts based on the reconstructed phase space. The Lyapunov exponents are nothing but a measure (in exponential scale) of the mean forecast errors using the nearest neighbour points in the phase space. However, when analysing a time series generated by a non-deterministic stochastic process, nothing guarantees the stability of the Lyapunov exponents. Oseledec's theorem only affects deterministic processes via ergodic theory. For a stochastic process, as the number of observations increases, the variability of the largest Lyapunov exponent will increase and, therefore, the largest Lyapunov exponent itself will also increase without limit with the sample size.

As we shall see, our simulations show an essential difference between chaotic and stochastic processes via Lyapunov exponents. If we want to reconstruct the trajectories of a time series in a phase space that is sampled from a stochastic process, there is no guarantee of convergence in any algorithm towards the largest Lyapunov exponent, because the Lyapunov exponents are not necessarily stable and independent of the initial conditions and sample size. For stochastic processes, the algorithm is only able to estimate *local Lyapunov exponents*. Local Lyapunov exponents are a measure of the local stability of the process and may be highly dependent on the sample size and the initial conditions.

Our simulations are based on different stochastic and chaotic processes. First of all, and following Barnett et al. (1997), let us consider samples of size 380 and 2000 observations of the following stochastic models

(i) A GARCH process of the following form:

$$y_t = h_t^{1/2} u_t ,$$

where h_t is defined by

$$h_t = 1 + 0.1 y_{t-1}^2 + 0.8 h_{t-1} ,$$

with $h_0 = 1$ and $y_0 = 0$.

(ii) A non-linear moving average (NLMA) process:

$$y_t = u_t + 0.8 u_{t-1} u_{t-2} .$$

(iii) An ARCH process of the following form:

$$y_t = (1 + 0.5 y_{t-1}^2)^{1/2} u_t ,$$

with the value of the initial observations set at $y_0 = 0$, and

(iv) An ARMA model of the form:

$$y_t = 0.8 y_{t-1} + 0.15 y_{t-2} + u_t + 0.3 u_{t-1} ,$$

with $y_0 = 1$ and $y_1 = 0.7$.

With these four stochastic models, the white noise disturbances u_t, are sampled independently from a standard normal distribution.

In order to provide further, and stronger, evidence supporting our claim that the observed invariance property of the largest Lyapunov exponent holds for all chaotic processes, we also consider the chaotic Feigenbaum recursion, the Hénon map and the Lorenz attractor:

(v) We have used a Feigenbaum recursion with parameter 4 where the map is fully chaotic, that is :

$$y_t = 4y_{t-1}(1 - y_{t-1}),$$

where the initial condition was set at $y_0 = 0.7$.

(vi) The Hénon (1976) map is described by the following system:

$$x_{t+1} = 1 - 1.4x_t^2 + y_t$$
$$y_{t+1} = 0.3x_t$$

with the initial conditions $x_0 = 0.5$ and $y_0 = 0.2$.

(vii) The well-known Lorenz (1963) attractor is the three-dimensional continuous-time system:

$$\dot{x} = 16(y - x)$$
$$\dot{y} = x(45.92 - z) - y.$$
$$\dot{z} = xy - 4z$$

with the initial conditions $x_0 = 0.2$, $y_0 = 0.4$ and $z_0 = 20$.

Lorenz´s system was solved using a straightforward fourth-order Runge-Kutta method, resulting a sampling period in the resolution of the system of approximately $\Delta t \cong 0.01$. Considering the average mutual information $I(T)$ for the signal $x(t)$ obtained after integration of the Lorenz´s system, the minimum of this function is at $T = 10$; following Abarbanel (1996), a time lag $\tau = 10$ was used in order to obtain a series $x(t_0 + n\tau)$, $n = 1, ..., 10.000$ as is usual for the phase reconstruction. The initial point was chosen near the attractor and transient points were discarded.

We calculated the largest Lyapunov exponent applying the algorithm proposed by Rosenstein *et al.* (1993) to the time series generated by these models for each sample size between 200 and 10.000 taking increments of 100 observations (i.e., 200, 300, 400, ...10.000). Figure 1 shows the results of estimating the largest Lyapunov exponents $\hat{\lambda}_{max}$ for the stochastic models used in Barnett *et al.* (1997) and for the three new chaotic series (Feigenbaum and Hénon maps and Lorenz attractor) for different sample sizes, from 200 to 10.000 observations, for a moving block of size *d*=5. Other moving block sizes show similar results.

[Figure 1]

Given the evidence presented in Figure 1, the existence of a positive largest Lyapunov exponent does not imply the presence of chaos in a given time series. However, Figure 1 does show an interesting and essential difference between chaotic and stochastic processes. While the largest Lyapunov exponent in the deterministic models stabilises (in some cases even slightly decreases) as the sample size increases, for all the stochastic processes, the largest Lyapunov exponent always increases with the sample size. The stability of $\hat{\lambda}_{max}$ with the sampling size for chaotic processes, versus the positive relationship for stochastic processes, appears to be an essential difference between chaotic and stochastic processes. This difference may be explained by Oseledec's (1968) theorem, which guarantees the stability of $\hat{\lambda}_{max}$ in chaotic processes, and by the infinite dimensionality of the noise present in stochastic processes. This behaviour recalls the well-known process of saturation of the correlation dimension in a chaotic time series when the embedding dimension increases. As a matter of fact, this is the base of the test proposed by Grassberger and Procaccia (1983) to detect deterministic chaos.

There exists an important observation with respect to the largest Lyapunov exponent estimations obtained in the Lorenz attractor (Figure 1). The numbers that we show in Figure 1 are not the largest Lyapunov exponents of the Lorenz attractor (for block sizes 5) because it is not possible to obtain these numbers without knowing the lag or reconstruction delay Δt that was used when sampling the time series from the continuous system. Observe that without Δt it is impossible to implement the expressions (1) or (2). This lag Δt is crucial because the Lyapunov exponents measure the rate at which system processes create or destroy information per time units $i\Delta t$ in expressions (1) and (2). So the exponents are expressed in bits of information per second. From this point of view, knowledge of the precise time units present in the time series is crucial in order to estimate the exact size of the Lyapunov exponents. Without this a priori information it is impossible to guess the precise time scale of the exponents obtained by Rosenstein *et al.* algorithm.

The numbers shown in Figure 1 are only fractions of the largest Lyapunov exponents. Observe that in this case λ_{max} is estimated through a time series obtained by a fourth-order Runge-Kutta method, where the sampling period of integration is of the order $\Delta t \approx 0.01$. In addition, the time series used for estimating λ_{max} is obtained by sampling the integrated trajectory with a time lag of $\tau = 10$. So the numbers of Figure 1 have to be multiplied by the factor $\frac{1}{\Delta t \tau} \approx 10$ in order to get the real scale of the largest Lyapunov exponent in the Lorenz system.

4. A new test for distinguishing chaos from random behaviour via Lyapunov exponents

In this section, we propose a new test, based on the stability of the largest Lyapunov exponent from different sample sizes, to detect chaotic dynamics in time series. As we will see, this new test is rather powerful when compared to different stochastic alternatives, both linear and non-linear.

This new test has a deterministic process as the null hypothesis, while the alternative hypothesis is that of a stochastic process (i. e., high-dimensional chaos), since randomness can be viewed as infinite-dimensional chaos.

In order to improve the estimation of the largest Lyapunov exponents and following Hastie *et al.* (2001), we will use a procedure known as "bagging" that is based on averaging bootstrap samples. Bootstrap aggregation or bagging, averages the estimations over a collection of bootstrap samples, thereby reducing the variance. Bagging can dramatically reduce the variance of unstable procedures, leading to improved estimations because averaging reduces variance and leaves bias unchanged, which will often decrease mean squared error. Given this, assume a time series of length *N*, $\left\{x_1, x_2, \ldots\ldots, x_N\right\}$. Let us divide the time series into different sub-samples, each one of which contains the precedent $\left\{x_1, x_2, \ldots, x_{T_1}, \ldots, x_{T_2}, \ldots, x_{T_{r-1}}, \ldots, x_{T_r} = x_N\right\}$, and consider an empirical distribution of the largest Lyapunov exponent from 100 moving block bootstraps of this time series for the different sub-samples $\left\{x_1, x_2, \ldots, x_{T_i}\right\}$, for $i=1,\ldots,r$. Therefore, our estimation of the dominant Lyapunov exponent for sample size T_i will be $\left\langle \lambda_{max}(T_i) \right\rangle$, that is, the mean of the distributions of the 100 largest Lyapunov exponents computed from those sample sizes T_i, which corresponds to the bootstrap aggregation or bagging of 100 bootstrap samples of the largest Lyapunov exponents.

Given that we have shown in Figure 1 that the largest Lyapunov exponent stabilises (or even decreases, as for example in the case of Hénon and Lorenz attractors) when increasing the sample size in a deterministic process, but it increases with the sample size in a stochastic process, we propose using $\left\langle \lambda_{max}(T_i) \right\rangle$ to test for the stability of the largest Lyapunov exponent. This non-increasing property of largest Lyapunov exponent with the sample size for chaotic processes may be tested recalling the traditional econometric test of linear independence between the bootstrap aggregation of the 100 largest Lyapunov exponents $\left\langle \lambda_{max}(T_i) \right\rangle$, in every sample size, and the sample size *T*. To that end, we have performed a linear regression of

$$\left\langle \lambda_{max}(T) \right\rangle = \alpha_0 + \alpha_1 T + \varepsilon_T \quad for \;\; T = T_1, \ldots, \;\; T_r = N, \tag{3}$$

so that the estimated parameter $\hat{\alpha}_1$ can be used to test if the largest Lyapunov exponent does not increase with sample size, implying an underlying deterministic process.

The null hypothesis H_0 and the alternative hypothesis H_1 are formulated as follows:

$H_0 : \; \alpha_1 \leq 0$ *(deterministic process)*

$H_1 : \; \alpha_1 > 0$ *(stochastic process)*

Observe that the alternative hypothesis of this test is closely associated with the estimation of λ_{max} through a direct method like Rosenstein's, where pure random processes have positive largest Lyapunov exponents (infinite in theory). Notice also that in order to implement this statistical test it is convenient to use the estimate of the asymptotic variance-covariance matrix of α_0 and α_1 proposed by Newey and West (1987) that is robust with respect to both heteroskedasticity and autocorrelation for the OLS estimations.

Finally, observe that under the null of a constant dependent variable and no stationary explanatory variables, the distribution of the implied test statistic has to be worked out and small sample properties need to be studied. So, in order to outperform the critical values of our test we have simulated 250 replications of the linear regression (3) under the null, obtaining an empirical distribution of the statistic α_1 / s_1, where s_1 is the standard deviation of α_1 in every simulation. The main problem in this Monte Carlo simulation is that we have more than one process in the null hypothesis of deterministic chaos; in simulating (3) the null would be represented by a huge amount of processes like Feigenbaum´s, Hénon´s, Lorenz ´s etc. So we have simulated in (3) a combination of the most well known possibilities for the null hypothesis that is 250 replications of every one of Feigenbaum, Hénon and Lorenz processes and the critical values have been taken from that composed empirical distribution. The critical values of the empirical distribution of the statistics α_1 / s_1 are displayed in Table 1 for different block sizes. The striking critical values corresponding to block size 2 and sample size equal to 2000 are not surprising because of the three dimensional nature of Lorenz´s attractor for which dynamics in two dimensions is poorly represented. In order to provide comparisons, in the last row of Table 1 we also give the critical values of the t-Student distribution.

[Table 1]

5. Applications

In this section we provide the applications of our test in two different sceneries.

On the one hand we have tested for deterministic chaos with the stochastic data of the single blind controlled competition available in Barnett *et al.* (1997) and for the most representative chaotic processes like Feigenbaum´s, Hénon´s and Lorenz´s. On the other hand we have tested for chaos in several exchange rate time series.

5.1 The power of the test against the most popular non-linear models

In order to demonstrate the power of our test compared to the most popular non-linear models, we tested for chaos using the simulated data from the seven models presented in the previous section. In all cases, the largest Lyapunov exponents were estimated using the algorithm proposed in Rosenstein *et al.* (1993).

Following Barnett *et al.* (1997), we compute our tests twice: for small samples of 380 observations and for large samples of 2000 observations.

For the 380 observations case, the sub-sample sizes are as follows:

$$T_1 = 200\,,\ T_2 = 220\,,\ T_3 = 240\,,\ T_4 = 260\,,\ T_5 = 280\,,$$
$$T_6 = 300\,,\ T_7 = 320\,,\ T_8 = 340\,,\ T_9 = 360\,,\ T_{10} = 380.$$

For the 2000 observations case, the sub-sample sizes are as follows:

$$T_1 = 1000,\ T_2 = 1020,\ T_3 = 1040,\ ....,\ T_{50} = 1980,\ T_{51} = 2000.$$

The key parameters in Rosenstein *et al.*'s algorithm have been selected, both for deterministic and stochastic processes, following the author's recommendations, as follows:

- The embedding dimension, which coincides with the moving-block length *d*, has been selected between two and six.
- The lag or reconstruction delay has been fixed to one in all cases. In the case of Lorenz's series, this is due to an a priori sampling lag of $\tau = 10$ which has been used in order to avoid significant autocorrelations between data.
- The mean period of the time series, which restricts nearest neighbours to having a temporal separation greater than the mean period, allows us to consider each pair of neighbours satisfying this constraint as being sufficiently close together given the initial conditions, for different trajectories. Following

Rosenstein *et al.*'s recommendations and his program MTRCHAOS 1.0, we estimated the mean period as the reciprocal of the mean frequency of the power spectrum.

On the other hand, consistently locating the region for extracting λ_{max} without a priori knowledge of the correct slope in the linear region is a delicate question, as several authors have pointed out. After a short transition, there is a long linear region that is used to extract the largest Lyapunov exponent. Implementing our test, the number of discrete-time steps allowed for divergence between nearest neighbours has been set at $i = 3$. The location of the linear region to extract λ_{max} proposed by Rosenstein *et al.* is necessarily visual and therefore difficult to reproduce in the bootstrapping framework. In this case our choice of the number of discrete-time steps allowed for divergence between nearest neighbours may produce a small bias in the estimation of λ_{max} with respect to original Rosenstein *et al.´s* paper. Nevertheless, these biases do not appear to be very significant in the implementation of our test.

Tables 2 to 3 show the results of our regression testing for the stability of the mean largest Lyapunov exponent $\langle \lambda_{max}(T) \rangle$ (for a sample of 100 largest Lyapunov exponents estimated by bootstrapping) when the sample size *T* increases. Table 2 corresponds to the small sample size of 380 observations and Table 3 to the large sample size of 2000 observations.

As can be seen, if the 1% marginal significance level is used, the test correctly distinguishes deterministic from random behaviour for sample sizes of 2000 observations (Table 3). The test also distinguishes correctly deterministic from random behaviour for sample sizes of 380 observations with the only exception of the ARMA process, for moving block size 5, and for NLMA, ARCH and ARMA processes, for moving block size 6, where the null is accepted incorrectly (Table 2).

[Table 2] and [Table 3]

Therefore, our simulation results suggest that our test correctly rejects chaos for the GARCH, NLMA, ARCH and ARMA stochastic processes in large sample sizes and for all moving block lengths, presenting occasional problems for rejecting the null in several stochastic processes for small sample sizes. On the contrary, our test accepts chaos in the case of the well-known chaotic processes for moving-block lengths from 2 to 6 for both sample sizes of 380 and 2000.

So our test works perfectly for large samples (2000 observations), although for small samples (380 observations) the test may present some imprecision. Bearing in mind the stochastic nature of the bootstrapping procedure used to estimate the mean of largest Lyapunov exponents $\langle \lambda_{max}(T) \rangle$ which represents the basis of our test, it appears to be convenient to obtain the size and power of our statistical test, for both sample sizes, in the different models considered above. To that end, we have generated the quantity $\langle \lambda_{max}(T) \rangle$ one hundred times in order to repeat the test, observing the percentage of times that the null hypothesis is rejected when it is correct (deterministic processes), and the percentage of times that the null hypothesis is accepted when it is false (stochastic processes). The size and power of our test in the different models are shown in Table 4 for the sample size of 380, and Table 5 for the sample size of 2000, for the nominal levels of significance of 90%, 95% and 99%.

[Table 4] and [Table 5]

As we can see in Table 4, with a sample size of 380, the power of our test against stochastic processes is near 1 for most of the stochastic models analysed, and for most moving block sizes, except for a few cases. There are only five exceptions: the NLMA model that fails on block sizes 4 and 6 for all levels; the ARCH model that fails on block size 6 for all levels, and the ARMA model that also fails on block sizes 5 and 6, for all levels. In order to construct Table 4 we have considered the first 380 observations of the available series.

In Table 4 we have also estimated the size of our test for a sample size of 380. The results are diverse and depend on the process and the block size. The variability of these results depends on the complex null hypothesis that entails the "chaos" which has obliged us to resort to a mixture of sample distributions in order to work out an empirical statistical test under the null of a non-stationary explanatory variable.

Table 5 shows the size and power of our test for the sample size of 2000. In this case our test emphatically rejects the stochastic processes and accepts adequately the chaotic processes with few exceptions. Also, the test would appear incorrectly sized in some cases due to the complex null hypothesis.

5.2 Evidence of deterministic chaos on financial data

In recent years several economic models have been developed, generating chaos in economic variables. Among many others we quote Grandmont (1985), who developed a model of a chaotic business cycle, and Brock and Hommes (1998) who provided a model with heterogeneous beliefs of agents that produces chaos in stock prices. Nevertheless, as Shintani and Linton (2003a) pointed out, looking for the empirical evidence of chaos in macroeconomic data is an elusive task because of the reduced sample sets available to the researchers. So, we concentrate our empirical research on financial time series, particularly on exchange rate series.

Financial markets are highly complex feedback systems. A considerable number of studies have suggested that, due to heterogeneity in expectations, structural non-linear financial models produce chaotic dynamics. For example, DeGrauwe *et al.* (1993) develop a sticky-price monetary model showing how the interaction between chartists and fundamentalists is capable of generating chaotic behaviour in exchange rates. Da Silva (2000) generalises the results in a more sophisticated framework. On the other hand, Szpiro (1994) argues that an intervening central bank may induce chaos in exchange rates.

There is substantial literature testing for non-linear dynamics and chaos on financial data due to the much greater amount of data available and to the superior quality of this data. Evidence of non-linearity on financial data is strong [see, for instance, Brock and Potter (1993) for a review]. Nevertheless, the evidence in favour of chaos is scant. Brock and Potter (1993) pointed out that the evidence consistent with chaos in studies such as Scheinkman and LeBaron (1989), Frank and Stengos (1988), Mayfield and Mizrach (1989), and others, may be produced by non-predictable, non-stationary or any other structure which is difficult to predict out of sample rather than chaos.

On the other hand, it is well known in the financial literature that several proxy variables for volatility, such as the powers of absolute returns $|r_t|^{\alpha}$, $\alpha = 0.5, 1, 1.5, 2, 2.5$ have significant positive serial correlation over long lags, although the returns themselves contain little serial correlation [see Taylor (1986) and Ding, Granger and Engle (1993)]

We have tested for deterministic chaos in three exchange rate series corresponding to the German mark, the Canadian dollar and the French franc, all against the US dollar. All the series go from 4th January 1971 to 31st December 1998.

With the critical values of the empirical distribution obtained in Table 1, we have tested for chaos in the logarithmic returns $r_t = log(S_t) - log(S_{t-1})$ for sample sizes of around 2000 observations (S_t stands for the exchange rate on date t), and we also test for chaos in the powers of absolute returns $|r_t|^{\alpha}$, $\alpha = 0.5, 1, 1.5, 2$.

Table 6 shows the exchange rates, periods and block sizes where our test accepts the null of deterministic chaos. On the one hand, our results accept the null hypothesis of chaos in the returns of the French franc/US dollar and in the returns of the Canadian dollar/US dollar series from January 1971 to June 1981, for block size 2 in both cases. The results are not robust to the block size because the null is rejected for higher block sizes. On the other hand, we have also detected deterministic chaos for all powers of absolute returns for the French franc/US dollar and the Canadian dollar/US dollar, for block size 2, during the same period where the chaos was detected for the return series. Again, the results are not robust to the block size.

For the German mark/US dollar, no chaotic behaviour is detected, neither in the return series nor in the powers of absolute returns in any block size.

According to Tables 4 and 5, where the power of our test against all stochastic processes is equal to one for block size 2, our results suggest some signs of deterministic chaos during the seventies. Nevertheless, these signs are weak because they are only detected for block size two.

6. Concluding remarks

Empirical research on the detection of chaotic behaviour has expanded rapidly, but results on small sample sizes have tended to be rather inconclusive, due to the lack of appropriate testing methods.

The problem of distinguishing chaotic from random behaviour is a very complex task limited by the number of observations available and the dimension of the chaotic attractor that generates the process. If the dimension of the attractor is large enough, the amount of data needed to test it may be prohibitive.

The general practice has been to take the existence of a positive Lyapunov exponent as an indication that the system is chaotic. However, this condition is not sufficient for the detection of chaos, and does not help us to distinguish a chaotic process from stochastic one. Indeed, any standard direct algorithm for calculating the largest Lyapunov exponent will find a finite, positive value for this exponent, both for chaotic and stochastic processes.

In this paper, we combine the bootstrap statistical framework for hypothesis testing using the computed Lyapunov exponents (Gençay, 1996), with the ergodic theory of deterministic dynamical systems in order to develop a new test to detect chaotic dynamics in time series. The new test is based on the stability of the mean of the distributions of the largest Lyapunov exponent estimated from different sample sizes, which is guaranteed by Oseledec's (1968) theorem. This theorem provides a strong feature of deterministic processes that is not shared by stochastic processes. We show that, while for stochastic processes (both linear and non-linear) the largest Lyapunov exponent increases with the sample size, for chaotic series the largest Lyapunov exponent is invariant when increasing the sample size. We compute the largest Lyapunov exponent using a robust version of the algorithm proposed by Rosenstein *et al.* (1993), considering the mean of divergences between pairs of neighbouring trajectories.

We have applied this new test to the simulated data used in the single-blind controlled competition among tests for non-linearity and chaos generated by Barnett *et al.* (1997), as well as several chaotic series, both for small and large samples (380 and 2000 observations, respectively). The results suggest that the new test has a high discriminatory power against interesting stochastic alternatives, both linear and non-linear (GARCH, NLMA, ARCH and ARMA).

Finally we have tested for deterministic chaos in three exchange rate series corresponding to the German mark, the Canadian dollar and the French franc, all against the US dollar. All the series go from 4 January 4 1971 to 31 December 1998. Signs of deterministic chaos have been detected for the returns and for all powers of absolute returns for the French franc/US dollar and the Canadian dollar/US dollar during the seventies. In contrast, no chaotic behaviour has been detected in any part of the sample for the the German mark/US dollar.

Signs found of chaotic dynamics could be associated with the long term fluctuations of the US dollar exchange rate during the seventies and eighties (see, e.g. Krugman, 1988). There are several reasons for these long term fluctuations such as irrational speculations (Westerhoff, 2003), heterogeneity of traders' expectations of a future currency exchange rate development (DeGrauwe *et al.*, 1993), exchange-rate intervention (Szpiro, 1994), expectations relating to future monetary policy (Federici and Santoro, 2001), as well as real economic shocks. Indeed, the period analysed presents several monetary changes of international impact that could lead to long term fluctuations in currency exchange rates.

Therefore, the results presented in this paper suggest that our test constitutes a contribution to the in-depth discussion existing in the literature. We provide an additional test for detecting low dimensional chaos that has the ability to distinguish between deterministic or stochastic processes, and proves to be very powerful when compared to different stochastic alternatives (both linear and non-linear) for both large and small sample sizes.

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Table 1: Critical values for the empirical distribution of α_1 / s_1 under the null of deterministic process $\alpha_1 \leq 0$ for the linear regression (1).

$$\langle \lambda_{max}(T) \rangle = \alpha_0 + \alpha_1 T + \varepsilon_T \quad for \quad T = T_1, \ldots, \ T_r = N$$

Sample size	T=380			T=2000		
	q(90%)	*q(95%)*	*q(99%)*	*q(90%)*	*q(95%)*	*q(99%)*
Block size=2	2.98485	3.19223	3.49616	6.03901	6.23596	6.54680
Block size=3	3.13403	3.57614	4.12994	-1.21598	-1.08813	-0.81998
Block size=4	1.75335	1.99139	2.54167	-3.71855	-3.24156	-2.44491
Block size=5	3.45828	3.70437	4.22816	-4.15120	-3.59367	-2.84705
Block size=6	7.36799	8.05090	9.96781	-0.67852	-0.51258	-0.20904
Critical t-values	1.3968	1.8595	2.8965	1.2996	1.6776	2.4077

Notes:

(1) The critical values of the statistic test for α_1 under the null of a constant dependent variable and a non-stationary explanatory variable, have been worked out studying the small sample properties.

Table 2: Test for the stability of the largest Lyapunov exponents for stochastic and chaotic processes.							
Sample size=380	GARCH	NLMA	ARCH	ARMA	Feigenbaum	Hénon	Lorenz
Coefficients of linear regression	$\hat{\alpha}_1$	$\hat{\alpha}_1$	$\hat{\alpha}_1$	$\hat{\alpha}_1$	$\hat{\alpha}_1$	$\hat{\alpha}_1$	$\hat{\alpha}_1$
Block size=2	0.00056 (23.62802[a])	0.00054 (5.51635[a])	0.00048 (20.62628[a])	0.00074 (6.28863[a])	0.00001 (0.50218)	0.00003 (1.63292)	0.00018 (2.61055)
Block size=3	0.00043 (10.71145[a])	0.00026 (5.22415[a])	0.00035 (7.38378[a])	0.00054 (5.73568[a])	0.00002 (0.39311)	-0.00004 (-1.17766)	0.00011 (1.64782)
Block size=4	0.00035 (9.22046[a])	0.00004 (2.75127[a])	0.00037 (6.96621[a])	0.00023 (3.07224[a])	0.00003 (1.02935)	0.00004 (0.98808)	0.00005 (0.69311)
Block size=5	0.00038 (8.27279[a])	0.00016 (8.63505[a])	0.00026 (5.64994[a])	0.00023 (3.79438)	0.00021 (2.75057)	-0.00001 (-0.17596)	0.00008 (1.27820)
Block size=6	0.00024 (13.45544[a])	0.00018 (6.73369)	0.00019 (9.03211)	0.00020 (4.62910)	0.00037 (5.00552)	-0.00003 (-0.74018)	0.00015 (3.55523)

Notes:

(1) OLS estimation of the linear regression $\langle \lambda_{max}(T) \rangle = \alpha_0 + \alpha_1 T + \varepsilon_T$ with t-ratio in brackets.

(2) [a] denotes rejection of the null hypothesis $H_0: \alpha_1 \leq 0$ *(deterministic process)* at the 1% level, following critical values in Table 1.

Table 3: Test for the stability of the largest Lyapunov exponents for stochastic and chaotic processes.							
Sample size=2000	GARCH	NLMA	ARCH	ARMA	Feigenbaum	Hénon	Lorenz
Coefficients of linear regression	$\hat{\alpha}_1$	$\hat{\alpha}_1$	$\hat{\alpha}_1$	$\hat{\alpha}_1$	$\hat{\alpha}_1$	$\hat{\alpha}_1$	$\hat{\alpha}_1$
Block size=2	0.00009 (13.20015[a])	0.00009 (29.46069[a])	0.00008 (32.82068[a])	0.00009 (27.21705[a])	-0.00001 (-5.95412)	-0.00001 (-9.49223)	0.00001 (6.08057)
Block size=3	0.00008 (23.45342[a])	0.00008 (47.99761[a])	0.00006 (19.62940[a])	0.00006 (27.24370[a])	-0.00001 (-5.77040)	-0.00001 (-4.46646)	-0.00000 (-1.06341)
Block size=4	0.00005 (19.20440[a])	0.00005 (36.26796[a])	0.00005 (20.18802[a])	0.00004 (22.57941[a])	-0.00000 (-3.25196)	-0.00001 (-5.33878)	-0.00001 (-11.34195)
Block size=5	0.00003 (10.79934[a])	0.00004 (13.31915[a])	0.00003 (20.62653[a])	0.00002 (12.79781[a])	-0.00000 (-3.79818)	-0.00001 (-6.7262)	-0.00001 (-6.48720)
Block size=6	0.00002 (18.13112[a])	0.00032 (13.19118[a])	0.00000 (24.22810[a])	0.00002 (13.30009[a])	-0.00000 (-2.30516)	-0.00001 (-0.67674)	-0.00001 (-2.90258)

Notes:

(1) OLS estimation of the linear regression $\langle \lambda_{max}(T) \rangle = \alpha_0 + \alpha_1 T + \varepsilon_T$ with t-ratio in brackets.

(2) [a] denotes rejection of the null hypothesis $H_0: \alpha_1 \leq 0$ *(deterministic process)* at the 1% level, following critical values in Table 1.

Table 4: Size and power of largest Lyapunov exponent test: sample size = **380**						
	Level	Block size=2	Block size=3	Block size=4	Block size=5	Block size=6
GARCH $\Pr(reject\ \ H_0/\alpha_1 > 0)$	90%	1.00000	1.00000	1.00000	1.00000	0.98400
	95%	1.00000	1.00000	1.00000	1.00000	0.92800
	99%	1.00000	1.00000	1.00000	1.00000	0.66400
NLMA $\Pr(reject\ \ H_0/\alpha_1 > 0)$	90%	1.00000	1.00000	0.55200	0.99600	0.03200
	95%	1.00000	1.00000	0.40400	0.99200	0.01200
	99%	1.00000	1.00000	0.19600	0.92400	0.00000
ARCH $\Pr(reject\ \ H_0/\alpha_1 > 0)$	90%	1.00000	1.00000	1.00000	1.00000	0.60000
	95%	1.00000	1.00000	1.00000	1.00000	0.40800
	99%	1.00000	1.00000	1.00000	0.99200	0.13200
ARMA $\Pr(reject\ \ H_0/\alpha_1 > 0)$	90%	1.00000	1.00000	1.00000	0.41600	0.00000
	95%	1.00000	1.00000	0.99600	0.12400	0.00000
	99%	1.00000	1.00000	0.72000	0.01200	0.00000
Feigenbaum $\Pr(accept\ \ H_0/\alpha_1 \le 0)$	90%	1.00000	0.98400	0.72000	0.72800	0.80800
	95%	1.00000	0.99200	0.85600	0.86800	0.93600
	99%	1.00000	1.00000	0.97200	0.97600	1.00000
Hénon $\Pr(accept\ \ H_0/\alpha_1 \le 0)$	90%	1.00000	1.00000	0.98400	1.00000	1.00000
	95%	1.00000	1.00000	0.99600	1.00000	1.00000
	99%	1.00000	1.00000	1.00000	1.00000	1.00000
Lorenz $\Pr(accept\ \ H_0/\alpha_1 \le 0)$	90%	0.70400	0.72000	1.00000	0.97600	0.89600
	95%	0.85200	0.86000	1.00000	0.98400	0.91600
	99%	0.97200	0.97200	1.00000	0.99600	0.97200

Table 5: Size and power of largest Lyapunov exponent test: sample size = **2000**						
	Level	Block size=2	Block size=3	Block size=4	Block size=5	Block size=6
GARCH	90%	1.00000	1.00000	1.00000	1.00000	1.00000
$\Pr(reject\ \ H_0/\alpha_1 > 0)$	95%	1.00000	1.00000	1.00000	1.00000	1.00000
	99%	1.00000	1.00000	1.00000	1.00000	1.00000
NLMA	90%	1.00000	1.00000	1.00000	1.00000	1.00000
$\Pr(reject\ \ H_0/\alpha_1 > 0)$	95%	1.00000	1.00000	1.00000	1.00000	1.00000
	99%	1.00000	1.00000	1.00000	1.00000	1.00000
ARCH	90%	1.00000	1.00000	1.00000	1.00000	1.00000
$\Pr(reject\ \ H_0/\alpha_1 > 0)$	95%	1.00000	1.00000	1.00000	1.00000	1.00000
	99%	1.00000	1.00000	1.00000	1.00000	1.00000
ARMA	90%	1.00000	1.00000	1.00000	1.00000	1.00000
$\Pr(reject\ \ H_0/\alpha_1 > 0)$	95%	1.00000	1.00000	1.00000	1.00000	1.00000
	99%	1.00000	1.00000	1.00000	1.00000	1.00000
Feigenbaum	90%	1.00000	1.00000	0.70400	0.70400	1.00000
$\Pr(accept\ \ H_0/\alpha_1 \leq 0)$	95%	1.00000	1.00000	0.85200	0.85200	1.00000
	99%	1.00000	1.00000	0.97200	0.97200	1.00000
Hénon	90%	1.00000	1.00000	1.00000	1.00000	0.70400
$\Pr(accept\ \ H_0/\alpha_1 \leq 0)$	95%	1.00000	1.00000	1.00000	1.00000	0.85200
	99%	1.00000	1.00000	1.00000	1.00000	0.97200
Lorenz	90%	0.70400	0.70400	1.00000	1.00000	1.00000
$\Pr(accept\ \ H_0/\alpha_1 \leq 0)$	95%	0.85200	0.85200	1.00000	1.00000	1.00000
	99%	0.97200	0.97200	1.00000	1.00000	1.00000

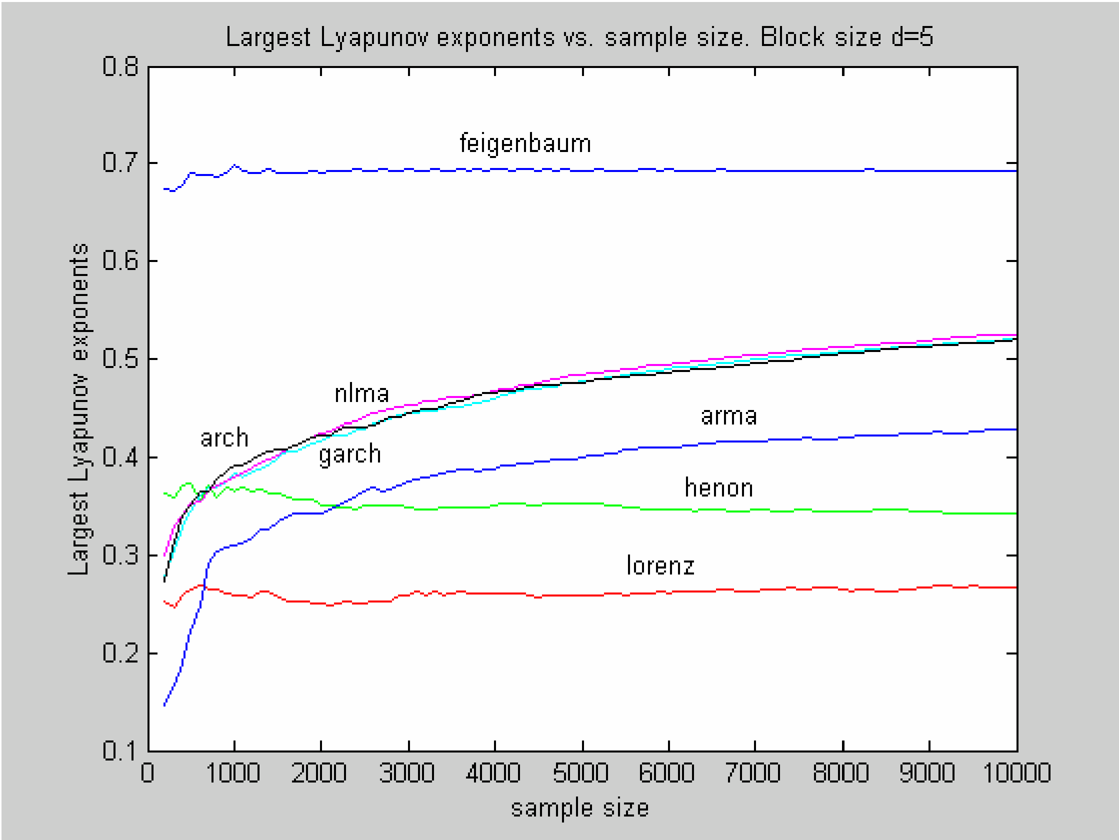
Table 6: Critical values for the empirical distribution of α_1 / s_1 under the null of deterministic process $\alpha_1 \leq 0$ for the linear regression (1). $\langle \lambda_{max}(T) \rangle = \alpha_0 + \alpha_1 T + \varepsilon_T \quad for \quad T = T_1, \ldots, \; T_r = N$		
Sample size	T=2000	
Block size=2	French-franc / USA-dollar From Jun 1973 to Jun 1981	Canadian-dollar / USA-dollar From Jun 1973 to Jun 1981
Returns: r_t	-0.000000567392 (-0.442773087510)	-0.000028974074 (-9.405226641112)
$\lvert r_t \rvert^{0.5}$	-0.000044848704 (-18.652187697785)	-0.000100484602 (-14.257695051415)
$\lvert r_t \rvert$	-0.000032639356 (-12.131668952056)	-0.000079231957 (-11.146644164362)
$\lvert r_t \rvert^{1.5}$	-0.000022395489 (-7.115107295481)	-0.000059246297 (-13.40553594015)
$\lvert r_t \rvert^{2}$	0.000005474017 (1.735723588500)	-0.000044532138 (-11.193154865891)

Notes:

(1) OLS estimation of the linear regression $\langle \lambda_{max}(T) \rangle = \alpha_0 + \alpha_1 T + \varepsilon_T$ with t-ratio in brackets.

(2) [a] denotes rejection of the null hypothesis $H_0 : \alpha_1 \leq 0$ *(deterministic process)* at the 1% level, following critical values in Table 1.

Figure 1


Largest Lyapunov exponents vs. sample size. Block size d=5
Largest Lyapunov exponents
sample size
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
1000
2000
3000
4000
5000
6000
7000
8000
9000
10000
feigenbaum
nlma
arch
garch
arma
henon
lorenz